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Complex spatial organization in a simple model of resource allocation

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Abstract. A dynamical model for the distribution of resources among competing agents is studied. The model is exactly solvable in the case of global competition, which leads to the accumulation of all the resources by the agent with the highest performance. On the other hand, local competition allows for a wider resource distribution, with a much weaker correlation with individual performances. Multiplicative processes give rise to almost-ordered spatial structures, through the enhancement of random fluctuations.

PACS. 87.23.Ge Dynamics of social systems – 87.23.Cc Population dynamics and ecological pattern formation – 05.65.+b Self-organized systems

1 Introduction

Though many real systems in the scopes of biology and the social sciences are well described by agent-based models with pair interactions [1–3], qualitatively similar to the physical description of interacting particles, a large class of biological and social processes are driven by interactions mediated by an external actor, which generally bears no resemblance with the individual agents. This kind of processes – which, in connection with physics, may be assimilated to the evolution of globally coupled dynamical systems [4] – is typically found in systems where agents compete for resources. A particularly relevant instance of this situation is given by competing biological species. In this case, in fact, the interaction is rarely given by struggling events between individuals of different species, but rather by accessing simultaneously to the limited resources provided by the environment. The performance of each agent, *i.e.* of each species, is here measured by its ability to get resources, which depends both on the capabilities of each individual and on global features such as the total population of the species.

The same scenario is found in some economical systems, for instance, in companies competing for financial resources, usually administrated by banks, or even in scientific research projects competing for funds from a government agency. A key ingredient in the dynamics of these systems is that the ability of each agent to get resources at a given time depends, often strongly, on the resources assigned to the agent at previous stages. This may give rise to a multiplicative process that, in the absence of buffer mechanisms, leads to resource accumulation by a single agent. In this paper, we consider a simple dynamical model that incorporates these elements and, in particular, we study the effects of competition at local level. The model with global competition and its exact solution are introduced in the next section. A few relevant variants of the model are also discussed. Then, we focus attention on local competition, which is found to soften the process of resource accumulation and to give rise, through the enhancement of fluctuations, to nontrivial spatial structures.

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2 The model. Global competition

Our system consists of an ensemble of N agents, each of them characterized by its productivity α_i. For future convenience, we consider the generic situation where all productivities are different. At each time step t, each agent is assigned a fraction $r_i(t)$ of the total available resources at that time. These resources are used to produce an amount of products $p_i(t) = \alpha_i r_i(t)$. At the next time step, resources are distributed among agents in amounts proportional to their production, namely,

$$r_i(t+1) = \frac{p_i(t)}{\sum_j p_j(t)} = \frac{\alpha_i r_i(t)}{\sum_j \alpha_j r_j(t)} \cdot \tag{1}$$

This equation can be fully solved for arbitrary productivities and initial conditions $r_i(0)$. The solution reads

$$r_i(t) = \frac{\alpha_i^t r_i(0)}{\sum_j \alpha_j^t r_j(0)} \cdot \tag{2}$$

Note that $\sum_i r_i(t) = 1$.

For long times, $r_i(t) \to 0$ for all i, except for the agent with the maximal productivity, $\alpha_{\max} = \max_i\{\alpha_i\}$, which receives all the available resources, $r_{\max}(t) \to 1$. Due to the multiplicative effect of resource allocation according to production, all resources are in the long run accumulated by the agent with the maximal productivity, giving rise to a sort of winner-takes-all state [3]. By analogy with population dynamics we say that the remaining agents become extinct. In fact, in connection with biological populations, this result is a realization of a well-known principle of ecology, namely, the principle of competitive exclusion [1,5]: in a system of biological species competing for the same resources, only one survives and the others undergo extinction. At moderately large t the resources assigned to each agent are well approximated by $r_i(t) = (\alpha_i/\alpha_{\max})^t r_i(0)/r_{\max}(0)$.

It is interesting to point out that in the late stages of the evolution, when $r_{\max} \approx 1$, equation (1) can be approximately written as

$$r_i(t+1) \approx \frac{a_i r_i(t)}{1+\sum_j a_j r_j(t)} \approx a_i r_i(t) \left[1 - \sum_j a_j r_j(t)\right], \tag{3}$$

with $a_i = \alpha_i/\alpha_{\max}$, and where now the sum does not include the term containing $r_{\max}(t)$. The last expression in equation (3) represents a multispecies Lotka-Volterra system [6] for competing agents. It evolves in discrete time steps, and can therefore be identified with a multispecies generalization of logistic evolution in the extinction regime.

Assuming that the productivities α_i in equation (1) are drawn at random from a distribution $P_\alpha(\alpha)$ and that, for simplicity, resources are evenly assigned at the beginning, $r_i(0) = N^{-1}$ for all i, the probability distribution for the individual resources at a given time is

$$P_r(r_i) = P_\alpha[\alpha_i(r_i)]\frac{\mathrm{d}\alpha_i}{\mathrm{d}r_i} = \frac{\alpha_{\max}}{r_i t} r_i^{1/t} P_\alpha\left(\alpha_{\max} r_i^{1/t}\right). \tag{4}$$

The dependence of this function on r_i through the power $r_i^{1/t}$ is very weak for large t, so that on a wide interval of the variable we find $P_r(r_i) \propto r_i^{-1}$. For long times, thus, resources are distributed among agents following a power law with exponent -1. Compare this result with Pareto's law of wealth distribution [3,7].

Figure 1 shows how the distribution $P_r(r)$ builds up as time elapses, in the numerical simulation of a 10^3-agent system. A power-law distribution develops at early stages, and the exponent -1 is already observed after a few tenth steps.

Model (1) admits several variations, that may be of interest in connection with the description of some real systems. For instance, the extinction of all but one agent can be avoided by modifying slightly the allocation method. Part of the total resources may be used to subsidize agents, in such a way that dropping of their resources below a certain level is prevented [8]. If at each time step a small fraction ρ of the total resources is evenly distributed among

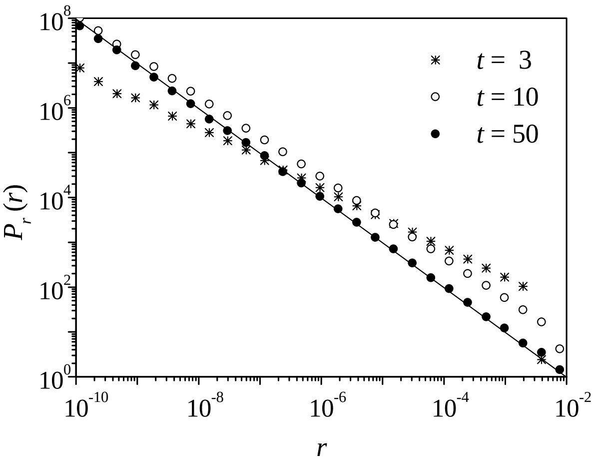

Fig. 1. Probability distribution for individual resources at three evolution stages, in a 10^3-agent system with a random uniform distribution of productivities in $(0,1)$. At $t = 0$ resources were evenly distributed among agents. Averages over 10^2 realizations are shown. The straight line has slope -1.

the agents, while the remaining is assigned according to production as above, the individual resources are always larger than ρ/N. Namely,

$$r_i(t+1) = \frac{\rho}{N} + (1-\rho)\frac{\alpha_i r_i(t)}{\sum_j \alpha_j r_j(t)} \cdot \tag{5}$$

In this situation, $P_r(r_i)$ becomes stationary for long times and, if ρ is small enough, the power-law dependence quoted above is found again for $r_i > \rho/N$.

A second variant consists in assuming that the productivity can in turn depend on the individual resources, for instance, as $\alpha_i(r_i) = \alpha_i^0 A(r_i)$. This additional nonlinearity is characterized by the function $A(r)$, which stands for the profile of the relation between productivity and resources. Taking $A(0) = 1$, the weight α_i^0 is the productivity for vanishing resources. If $A(r)$ increases with r, the winner-takes-all effect is enhanced.

The opposite case, where $A(r)$ is a decreasing function, is more interesting. Lower productivities for higher resources may be the consequence of size effects, crowding, or loss of efficiency due to less competition. In this case, the system evolves to a stationary situation where the agents whose productivity weights α_i^0 are above a certain threshold $\alpha_{\min}^0$ receive nonzero resources, whereas the agents with $\alpha_i^0 < \alpha_{\min}^0$ become extinct. In the stationary situation, all the surviving agents have the same productivity α. The values of $\alpha_{\min}^0$ and α can be obtained from the equation

$$\sum_i r_i = \sum_i A^{-1}(\alpha/\alpha_i^0) = 1, \tag{6}$$

where the sum runs over the surviving agents only, and A^{-1} is the inverse of the function A. Figure 2 shows the asymptotic values of r_i as a function of the weights α_i^0 for the case $A(r) = 1 - \beta r$, for three values of β. These

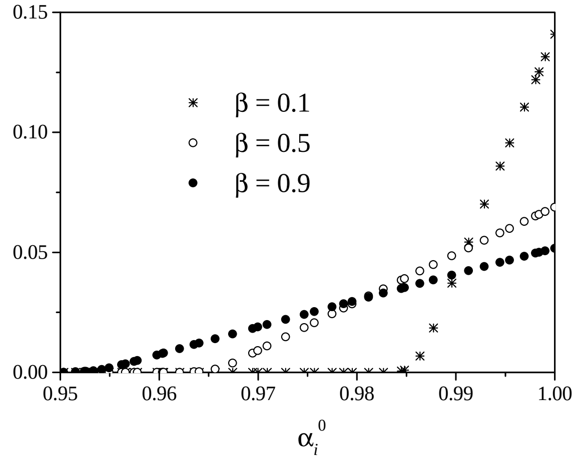

Fig. 2. Asymptotic resources r_i as a function of the weights α_i^0, for resource-dependent productivities of the form $\alpha_i(r_i) = \alpha_i^0(1-\beta r_i)$, and three values of β. The system consists of 10^3 agents, whose productivity weights α_i^0 have a random uniform distribution in $(0,1)$.

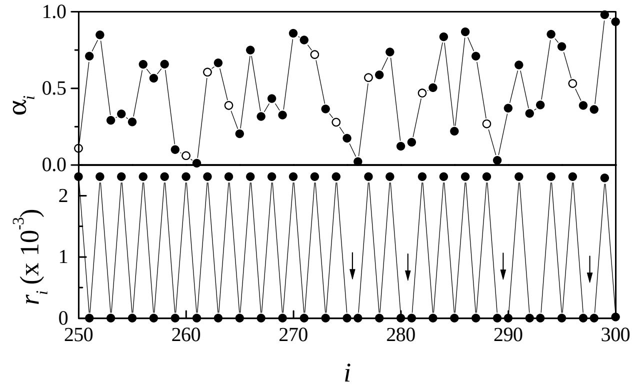

Fig. 3. Upper plot: Productivities α_i of 50 contiguous agents in a one-dimensional array of 10^3 agents. Lower plot: Asymptotic resources r_i for the same agents. Arrows indicate defects in the emerging periodic structure. Empty dots in the upper plot correspond to surviving agents whose productivity is *not* a local maximum.

numerical results correspond to single realizations of a 10^3-agent system where α_i^0 is chosen at random from a uniform distribution in $(0,1)$. Note than only the right end of this interval is shown in the plot.

The third instance, where $A(r)$ is a nonmonotonous function, may give origin to multiple stationary nontrivial solutions.

3 Local competition

In the following we focus the attention on a variant of model (1) that introduces a spatial distribution of agents. Though the total resources are still allocated among all the ensemble, the agents compete at a local level only. The set of agents that compete with a given agent i defines its neighbourhood $\mathcal{N}_i$. Resources are assigned according to

$$r_i(t+1) = \frac{1}{Z(t)} \frac{\alpha_i r_i(t)}{\sum_{j\in\mathcal{N}_i} \alpha_j r_j(t)}, \tag{7}$$

where

$$Z(t) = \sum_i \frac{\alpha_i r_i(t)}{\sum_{j\in\mathcal{N}_i} \alpha_j r_j(t)} \tag{8}$$

is a normalization factor that insures that $\sum_j r_j(t) = 1$ for all t. We have performed numerical simulations of this system in ensembles of $N = 10^3$ to 10^4 agents distributed on various geometries. The productivities α_i were drawn at random from a uniform distribution in $(0,1)$. Note that, in equations (1) and (7), a homogeneous rescaling of all productivities leaves the models invariant. Several distributions of initial conditions were tested, but there are no essential differences with the case where, at the first step, resources are evenly allocated, $r_i(0) = N^{-1}$. Therefore, we concentrate our simulations on this simple case.

In the first place, we consider a one-dimensional array of agents with periodic boundary conditions. The neighbourhood of the ith agent consists of its two nearest neighbours. We find that, for long times, approximately half of the agents become extinct and resources are evenly allocated among the surviving agents. Practically everywhere along the array agents are ordered in an alternating sequence of extinct agents and survivors. Occasionally, one finds two neighbour sites where both agents are extinct, but two survivors are never contiguous. In other words, in the neighbourhood of a surviving agent no other survivor can be found (*cf.* model (1)). The surprising feature of this asymptotic distribution is that the strong correlation between survival and productivity found for equation (1) is lost in equation (7). In fact, productivities are distributed completely at random along the array, whereas the resulting distribution of survivors is highly ordered. This is illustrated in Figure 3, where a 50-site portion of a 10^3-agent system is displayed. The upper plot shows a particular realization of the productivities α_i, and the lower plot shows the corresponding asymptotic resources r_i allocated to each agent. Extinct and surviving agents alternate in a practically regular structure, with a few defects marked by arrows. On the other hand, the productivity changes randomly from agent to agent so that, for instance, no apparent correlation is found between local maxima of productivity and the presence of survivors. Each empty dot in the upper plot, in fact, stands for a survivor one of whose extinct neighbours had a larger productivity. Practically one half of the survivors belong to this class. Surviving agents can even correspond to local minima of productivity.

The same features are found in a two-dimensional array with periodic boundary conditions, where the neighbourhood of each site consists of the four nearest neighbours. Figure 4 displays the asymptotic state on a (100×100)-site lattice ($N = 10^4$), where the full squares represent survivor sites. Regular domains where survivors

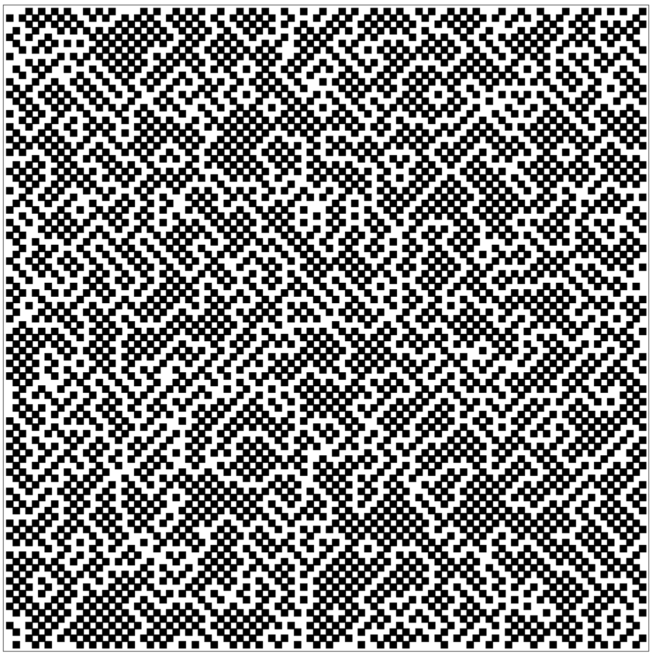

Fig. 4. Asymptotic state on a 100×100 - site lattice with nearest-neighbour competition and periodic boundary conditions. Full squares represent survivors.

and extinct agents alternate in both dimensions, separated by worm-like boundaries formed by extinct agents, are apparent. We have also verified that different definitions of the neighbourhood of a site, both in one and in two-dimensional lattices, produce similar almost-periodic asymptotic structures. Again, resources are evenly distributed among the surviving agents and, in neighbourhood of a survivor, no other survivor can be found. For instance, in a one-dimensional array where the neighbourhood consists of four sites (nearest and next-to-nearest neighbours), the resulting structure is a periodic sequence of two extinct agents and one survivor. Occasional defects, with larger zones of extinction, are also found.

In order to trace the origin of the almost-periodic structures emerging at asymptotically long times we have inspected the successive states of the system from the earliest stages, for the specific case of a one-dimensional array of agents with two neighbours per site and periodic boundary conditions. As above, productivities are drawn at random from a uniform distribution and $r_i(0) = N^{-1}$ for all i. We find that even at the first step of the evolution the distribution of resources already exhibits an almost-periodic sequence of relatively high and low values, in spite of the fact that productivities are spatially uncorrelated. Namely, for a substantial fraction of agents, we have $[r_{i+1}(1) - r_i(1)][r_i(1) - r_{i-1}(1)] < 0$. This observation reveals that the simple dynamics of model (7) is able to introduce strong spatial correlations to an initially uncorrelated state, even at the level of a single evolution step.

To quantify this effect, we take a set of five uncorrelated random numbers $\alpha_1, \ldots, \alpha_5$ drawn from the productivity distribution, and consider the combinations

$$\sigma_i = \frac{\alpha_i}{\alpha_{i-1} + \alpha_i + \alpha_{i+1}} \tag{9}$$

for $i = 2, 3, 4$. If the numbers $\alpha_1, \ldots, \alpha_5$ are identified with the productivities of a group of five contiguous agents, the quantities σ_i are proportional to the resources received at $t = 1$ by the three central agents in the group (*cf.* Eq. (7)). Numerical realizations of these quantities, over 10^6 random choices of $\alpha_1, \ldots, \alpha_5$, show that $(\sigma_4 - \sigma_3)(\sigma_3 - \sigma_2) < 0$ with probability $p_1 = 0.762$. Consequently, more than 76% of the agents are expected to belong to a periodic sequence of alternating high and low resources at the first time step. The remaining 24% stands for defects in the periodic structure.

The same kind of analysis can be performed for successive steps in the evolution. To calculate the resources allocated to three contiguous agents at time t, it is necessary to consider the productivities of a group of $3 + 2t$ agents. Following this scheme, we find that correlations are further enhanced by the dynamics. At the second evolution step, for instance, the probability for an agent to belong to the periodic structure grows to $p_2 = 0.782$. For later stages, we successively find $p_5 = 0.816$, $p_{10} = 0.839$, $p_{20} = 0.854$, and $p_{50} = 0.864$. These probabilities saturate at $p_\infty \approx 0.865$. Note that, as a byproduct, this calculation predicts a finite density of defects in the almost-periodic structure.

While, as discussed above, the emergence of almost-periodic patterns seems to indicate that no correlations subsist in model (1) between survival and productivity, inspection of the evolution equations as well as of equation (9) suggests that some remaining correlation should however exist. In fact, low productivities still imply few resources, so that the minima in the periodic structure should preferably coincide with sites with small α_i. Moreover, defects in the periodic structures – where extinct agents are found in consecutive sites – should also correspond to low-productivity zones. This is clearly illustrated by Figure 3, near $i = 275$ and 290.

Such remaining correlations can be characterized by the probability of survival as a function of the productivity, $p_s(\alpha)$. This probability is defined as the fraction of agents with productivity α which survive at asymptotically long times. We have numerically measured $p_s(\alpha)$, in series of 100 independent realizations of the productivity distribution for ensembles with $N = 10^3$, for various geometries and neighbourhoods. Figure 5 shows the survival probability as a function of α for one-dimensional arrays with different numbers of neighbours. Here, the correlation between survival probability and productivity is apparent. Agents with larger productivities are more likely to survive that those with small α. Note, however, that for $\alpha = 1$ the survival probability is less than unity, so that even with the maximal productivity there are chances of undergoing extinction. Conversely, for $\alpha \to 0$ (but $\alpha \neq 0$), the survival probability is finite. As the number of neighbours grows the probability is more concentrated towards

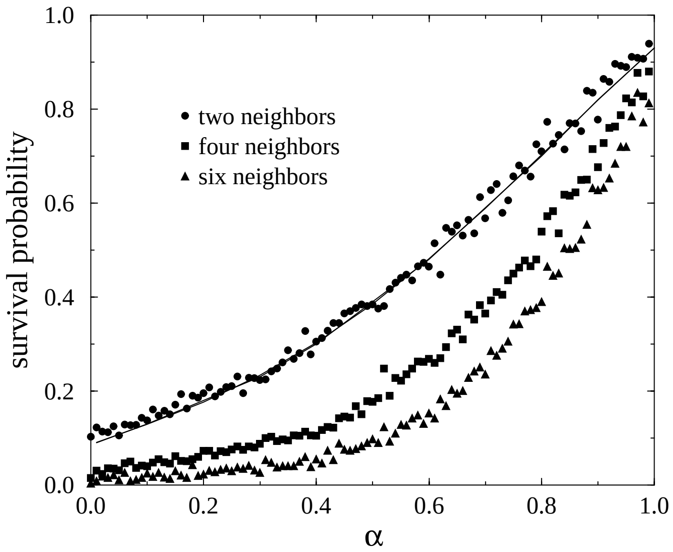

Fig. 5. Survival probability as a function of the productivity α for one dimensional arrays of 10^3 agents and different neighbourhoods. Dots correspond to average measurements over 100 realizations. The line shows the result of an independent calculation method (see text), for the case of two neighbours.

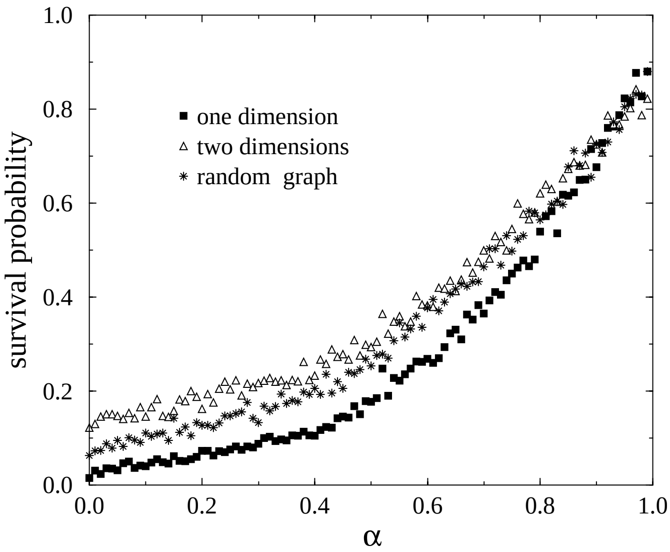

Fig. 6. Survival probability as a function of the productivity α for different geometries with four neighbours per site.

larger productivities, as expected. In the limit where the neighbourhood extends to the whole array the original model (1) is recovered, and the survival probability must vanish except for $\alpha = 1$.

The survival probability can be calculated independently of the numerical realization of the model, using expressions as in equation (9). For a fixed value of α_3 and different random choices of α_1, α_2, α_4 and α_5, an estimate at the first evolution step of the survival probability of the agent with productivity α_3 is given by the fraction of instances for which $\sigma_3 > \sigma_2, \sigma_4$. Generalizing this procedure for successive time steps, the estimate can be improved by considering later stages in the evolution. The line in Figure 5 corresponds to this estimate at $t = 50$. It shows an very good agreement with numerical realizations.

To compare now several geometries, we have measured the survival probability in systems where each agent has the same number of neighbours – specifically, four – on a one-dimensional array (nearest and next-to-nearest neighbours), a two-dimensional lattice (nearest neighbours) and a random graph with four connections per site. The results, shown in Figure 6, corresponds to averages over 100 realizations in systems with $N = 1000$ for one dimension and random graphs. In this latter case, the graph topology is chosen anew at each realization. For two dimensions, the results correspond to 10 realizations on a (100×100)-array. Though the general trend of $p_s(\alpha)$ is qualitatively the same for the three cases, some systematic differences are apparent. For instance, the survival probability in one dimension is appreciably lower than in the other geometries for low productivities. In this range, remarkably, the data for random graphs lies between those for one and two dimensions. For $\alpha > 0.8$, instead, the values of $p_s(\alpha)$ are hardly distinguishable within our numerical precision.

4 Summary and concluding remarks

We have here studied a simple model of resource allocation among competing agents. Competition is driven by the relative ability of agents to get resources, rather than by their mutual interaction. This ability is in turn determined by the performance of each agent in the use of the assigned resources, *i.e.*, by its individual productivity. The model, which in a suitable limit can be identified with a discrete-time Lotka-Volterra system for many competing species, can be exactly solved in the case where competition is global. This solution describes the progressive accumulation of resources by the agent with the largest productivity. During the transient, resources become distributed among agents following a power-law with a well defined exponent.

The allocation criterion admits several variants including, for instance, subsidizing mechanisms that prevent the extinction of most agents, or saturation effects due to resource-dependent productivity. We have focused attention on the variant where global competition is replaced by a local interaction, where each agent still has access to the whole amount of available resources but competes with its neighbours only. The strong, deterministic correlation between survival and maximal productivity that characterizes the model with global competition is replaced, in the model with local competition, by a much weaker, probabilistic correlation. Agents with high productivity are more likely to survive but, even with the maximal productivity, extinction cannot be completely discarded. On the other hand, agents with very low productivity have a chance of survival. This conclusion should be relevant to the possible applications of the present models, both to economy and to biology.

The loss of the above mentioned deterministic correlation is accompanied by the formation of an almost-regular spatial structure, with a periodic alternation of survivors and extinct agents. This structure is explained by the emergence of strong spatial correlations out of the fully uncorrelated distribution of productivities, due to the very action of the evolution rules. Nontrivial correlations in the conditional probabilities for certain combinations of uncorrelated variables – a rather simple but counterintuitive phenomenon – have recently been pointed out for series of random numbers [9]. The present model illustrates the occurrence of the same kind of phenomenon in a spatially extended dynamical system.

Further variants of the model deserve consideration in the future. We mention, in particular, the introduction of noise in the dynamics, as a representation of unavoidable fluctuations in individual and environmental conditions. Such fluctuations could have a nontrivial effect if, in turn, they depend on the individual resources. This is in fact expected if the mechanism by which each agent generates its products is thought of as a superposition of several processes, each of them using a certain portion of the individual resources and being affected by independent fluctuations. Resource-dependent fluctuations may be represented by multiplicative noises [8,10], which would contribute to the inherently multiplicative dynamics of the deterministic model analyzed here.

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